

Perturbative approach to diatomic lattices

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Abstract

By using a small rotation approach, we show that it is possible to obtain well behaved perturbed solutions for the amplitude of the electromagnetic field propagating in a photonic waveguide array.

1 Introduction

The analogy between linear lattices and atom-field interactions [1] has been a fundamental step for the emulation, via classical interactions, of quantum mechanical systems. This is not only important because of pure scientific reasons but also because of the possible applications in quantum information processing. In this latter case, the properties of classical systems have been used to realize quantum computational operations by quantum-like systems and, in particular, it has been show how a controlled-NOT gate may be generated in nonhomogeneous optical fibers [2]. At a fundamental level, *e.g.*, it has been possible to the emulate the most basic atom-field interaction, the Jaynes-Cummings model, theoretically [3] and experimentally [4] with arrays of photonic waveguides; and, just to give another example, it has been proposed to model non-linear coherent states [5] in waveguide arrays [6]. Linear coherent states have also been modeled via linear arrays of photonic waveguides [7].

2 Diatomic waveguide arrays

In a diatomic photonic waveguide the field amplitude in each way is obtained form the infinite system of coupled ordinary first order differential equations [3]

$$i \frac{du_n}{dz} - \omega(-1)^n u_n - \alpha(u_{n+1} + u_{n-1}) = 0, \quad n = -\infty, \dots, \infty, \quad (1)$$

where α and ω are arbitrary constants. The field at $z = 0$ may be given, in general by $u_n(z = 0) = \psi_n$, for which we only ask to be normalized. One can associate with this system a Schrödinger-like equation

$$i \frac{d|\psi(z)\rangle}{dz} = \hat{H}|\psi(z)\rangle, \quad (2)$$

subject to the *initial condition* $|\psi(z = 0)\rangle = \sum_{m=-\infty}^{\infty} \psi_m |m\rangle$. The "Hamiltonian" of this Schrödinger-like equation is given by

$$\hat{H} = \omega(-1)^{\hat{n}} + \alpha(\hat{V} + \hat{V}^\dagger), \quad (3)$$

and the linear operators \hat{V} and \hat{V}^\dagger are defined as

$$\hat{V} \equiv \sum_{n=-\infty}^{\infty} |n\rangle\langle n+1|, \quad \hat{V}^\dagger \equiv \sum_{n=-\infty}^{\infty} |n+1\rangle\langle n|, \quad (4)$$

acting over the vector space generated by the complete and orthonormal set $\{|n\rangle; n = -\infty, \dots, +\infty\}$.

If we propose as solution of the Schrödinger-like equation (2) the expansion

$$|\psi(z)\rangle = \sum_{k=-\infty}^{\infty} u_k(z) |k\rangle, \quad (5)$$

we obtain for the coefficients $u_n(z) = \langle n|\psi(z)\rangle$ the infinite system of first order ordinary differential equations (1). Thus, by solving the Schrödinger-like equation 2 means to solve the system 1.

The action of the operators \hat{V} and \hat{V}^\dagger on the states $|n\rangle$ is

$$\hat{V} |n\rangle = |n-1\rangle \quad \text{and} \quad \hat{V}^\dagger |n\rangle = |n+1\rangle. \quad (6)$$

3 Exact solution

The formal solution to the Schrödinger-like equation is

$$|\psi\rangle = e^{-iz\hat{H}} |m\rangle, \quad (7)$$

where the Hamiltonian \hat{H} is given by (3) and where we have already used the initial condition $|\psi(0)\rangle = |m\rangle$.

Because of the commutation relations of the operators involved in the Hamiltonian is not simple, one can not give a closed form for the evolution operator. In order to give a solution, we will make a change of basis, we will go from the discrete basis $\{|n\rangle; n = -\infty, \dots, \infty\}$ to a continuous basis defined by (the Fourier series)

$$|\phi\rangle = \sum_{n=-\infty}^{\infty} e^{in\phi} |n\rangle. \quad (8)$$

The inverse transformation is clearly

$$|n\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-in\phi} |\phi\rangle. \quad (9)$$

Therefore, the solution to the Schrödinger-like equation can be written as

$$|\psi\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-im\phi} e^{-iz\hat{H}} |\phi\rangle. \quad (10)$$

Thus, we have to analyze only the last part of the above equation, the action of the operator $e^{-it\hat{H}}$ on $|\phi\rangle$. We write the definition of the exponential operator, and after we split the even and odd powers,

$$\begin{aligned} e^{-izH} |\phi\rangle &= \sum_{k=0}^{\infty} \frac{(-iz)^{2k}}{(2k)!} \hat{H}^{2k} |\phi\rangle + \sum_{k=0}^{\infty} \frac{(-iz)^{2k+1}}{(2k+1)!} \hat{H}^{2k+1} |\phi\rangle \\ &= \sum_{k=0}^{\infty} \frac{(-iz)^{2k}}{(2k)!} \hat{H}^{2k} |\phi\rangle + \sum_{k=0}^{\infty} \frac{(-iz)^{2k+1}}{(2k+1)!} \hat{H}^{2k} \hat{H} |\phi\rangle. \end{aligned} \quad (11)$$

We have to study now the action of the Hamiltonian \hat{H} and the square of the Hamiltonian \hat{H}^2 on the wave functions $|\phi\rangle$. Using the definition (4), the fact that the operators \hat{V} and \hat{V}^\dagger commute, and that they anticommute with the operator $(-1)^{\hat{n}}$, it is easy to show that

$$\hat{H} |\phi\rangle = \omega |\phi + \pi\rangle + 2\alpha \cos \phi |\phi\rangle, \quad (12)$$

$$\hat{H}^2 |\phi\rangle = \Omega^2 |\phi\rangle, \quad (13)$$

and

$$\hat{H}^2 |\phi + \pi\rangle = \Omega^2 |\phi + \pi\rangle, \quad (14)$$

where

$$\Omega(\phi) = \sqrt{\omega^2 + 4\alpha^2 \cos^2 \phi}. \quad (15)$$

With all these results, we get back to equation (11), to obtain

$$e^{-iz\hat{H}} |\phi\rangle = \cos(\Omega z) |\phi\rangle - i2\alpha \cos \phi \frac{\sin(\Omega z)}{\Omega} |\phi\rangle - i\omega \frac{\sin(\Omega z)}{\Omega} |\phi + \pi\rangle. \quad (16)$$

Using now expression (10), we get

$$|\psi\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-im\phi} \left[\cos(\Omega z) |\phi\rangle - i2\alpha \cos \phi \frac{\sin(\Omega z)}{\Omega} |\phi\rangle - i\omega \frac{\sin(\Omega z)}{\Omega} |\phi + \pi\rangle \right].$$

To obtain the solution to the infinite system of differential equations, we recall that by definition $u_n = \langle n | \psi \rangle$, so

$$\begin{aligned} u_n(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-im\phi} \\ &\times \left\{ \cos[\Omega(\phi)z] \langle n | \phi \rangle - i2\alpha \cos \phi \frac{\sin[\Omega(\phi)z]}{\Omega(\phi)} \langle n | \phi \rangle - i\omega \frac{\sin[\Omega(\phi)z]}{\Omega(\phi)} \langle n | \phi + \pi \rangle \right\} \end{aligned} \quad (17)$$

It is very easy to prove that $\langle n|\phi\rangle = e^{in\phi}$, and substituting this in the above expression, we get the final answer to our problem,

$$u_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-i(m-n)\phi} \quad (18)$$

$$\times \left\{ \cos[\Omega(\phi)z] - i2\alpha \cos\phi \frac{\sin[\Omega(\phi)z]}{\Omega(\phi)} - (-1)^n i\omega \frac{\sin[\Omega(\phi)z]}{\Omega(\phi)} \right\}.$$

As the waveguide array is symmetric and infinite, we do not loose generality at all if we consider that the "central" waveguide is shined ($m = 0$). If the initial state is $|m\rangle = |0\rangle$, the result is reduced to

$$u_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\phi} \left\{ \cos[\Omega(\phi)z] - i2\alpha \cos\phi \frac{\sin[\Omega(\phi)z]}{\Omega(\phi)} - (-1)^n i\omega \frac{\sin[\Omega(\phi)z]}{\Omega(\phi)} \right\} d\phi. \quad (19)$$

Using the parity properties of the functions involved, we can finally write the solution of our original infinite system (1), when $m = 0$, as

$$u_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi) \left\{ \cos[\Omega(\phi)z] - i[2\alpha \cos\phi + (-1)^n \omega] \frac{\sin[\Omega(\phi)z]}{\Omega(\phi)} \right\} d\phi \quad (20)$$

Note that this solution satisfies the initial conditions. Indeed, if $z = 0$ we get $u_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi) d\phi$ that it is equal to $\delta_{n,0}$. Also note that if there is no interaction between the guides; i.e., $\alpha = 0$, we obtain

$$u_n(z) = e^{-i(-1)^n \omega z} \delta_{n,0} \quad (21)$$

that it is the solution of the trivial system of equations obtained.

In Figure 1, we present the behaviour of the exact solution of the guides 0 to 10, with $\omega = 1$ and $\alpha = 0.3$ for z from 0 to 100. For symmetry reasons the behaviour of the guides -1 to -10 is exactly the same.

4 Small rotation

We consider the case in which the parameters obey $\alpha \ll \omega$, and transform the Hamiltonian

$$\hat{H} = \omega(-1)^{\hat{n}} + \alpha(\hat{V} + \hat{V}^\dagger), \quad (22)$$

via the *unitary* transformation

$$\hat{R} = e^{\frac{\alpha}{2\omega}(-1)^{\hat{n}}(\hat{V} + \hat{V}^\dagger)}, \quad \hat{R}^\dagger = e^{-\frac{\alpha}{2\omega}(-1)^{\hat{n}}(\hat{V} + \hat{V}^\dagger)}, \quad (23)$$

such that $\hat{H}_R = \hat{R}\hat{H}\hat{R}^\dagger$. We use the relation $e^{s\hat{A}}\hat{B}e^{-s\hat{A}} = \hat{B} + s[\hat{A}, \hat{B}] + \frac{s^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \dots$ and note that we can cut the series to second order because $\alpha \ll \omega$ [9], obtaining

$$\hat{H}_R \approx \omega(-1)^{\hat{n}} + \frac{\alpha^2}{2\omega}(-1)^{\hat{n}}(\hat{V} + \hat{V}^\dagger)^2. \quad (24)$$

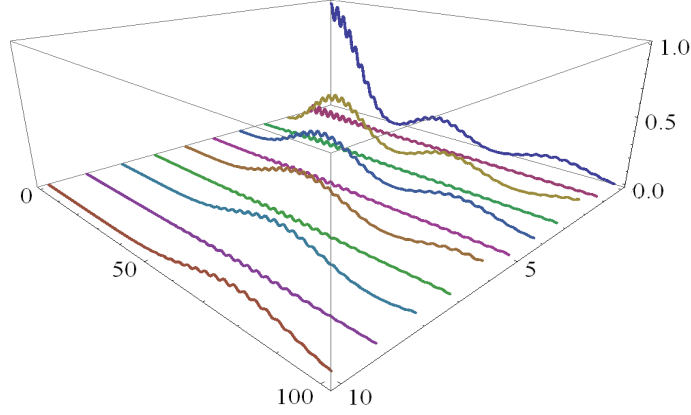


Figure 1: Exact solution of the guides 0 to 10 (for symmetry reasons the behavior of the guides -1 to -10 is exactly the same) with $\omega = 1$ and $\alpha = 0.3$ for z from 0 to 100

Because the parity operator $(-1)^{\hat{n}}$ and the operator $(\hat{V} + \hat{V}^\dagger)^2$ commute, we can obtain the propagator operator, $\hat{U}_R(z) = \exp(-i\hat{H}_R z)$, as the product

$$\hat{U}_R(z) = e^{-i\omega z(-1)^{\hat{n}}} e^{-i\frac{\alpha^2 z}{2\omega}(-1)^{\hat{n}}(\hat{V} + \hat{V}^\dagger)^2} \quad (25)$$

and the propagator associated to (22) is then

$$\hat{U}(z) \approx \hat{R}^\dagger e^{-i\omega z(-1)^{\hat{n}}} e^{-i\frac{\alpha^2 z}{2\omega}(-1)^{\hat{n}}(\hat{V} + \hat{V}^\dagger)^2} \hat{R}. \quad (26)$$

Expanding the square in the exponential, using once more the commutativity between $(-1)^{\hat{n}}$ and the squared \hat{V} operators and after some trivial algebra, we can write the ket $|\psi(z)\rangle$ as

$$|\psi(z)\rangle = \hat{R}^\dagger e^{-iz\left(\omega + \frac{\alpha^2}{\omega}\right)(-1)^{\hat{n}}} e^{-i\frac{\alpha^2 z}{2\omega}(-1)^{\hat{n}}[\hat{V}^2 + (\hat{V}^\dagger)^2]} \hat{R}|\psi(0)\rangle, \quad (27)$$

with $|\psi(0)\rangle$ the initial condition related with the waveguide that is shined. As $i(-1)^{\hat{n}}(\hat{V}^\dagger)^2 = \left[-i(-1)^{\hat{n}}\hat{V}^2\right]^{-1}$, we may develop the second exponential above in terms of Bessel functions by using their generating function and obtain

$$|\psi(z)\rangle = \sum_{k=-\infty}^{\infty} (-i)^k J_k\left(\frac{\alpha^2 z}{\omega}\right) \hat{R}^\dagger \hat{V}^{2k} e^{-iz\left(\frac{\omega^2 + \alpha^2}{\omega}\right)(-1)^{\hat{n}}} [(-1)^{\hat{n}}]^k \hat{R}|\psi(0)\rangle. \quad (28)$$

Using that $(-1)^{\hat{n}}\hat{V}^\dagger = -\left[(-1)^{\hat{n}}\hat{V}\right]^{-1}$, again the properties of the V operators and the generating function of the Bessel functions, we can write the following

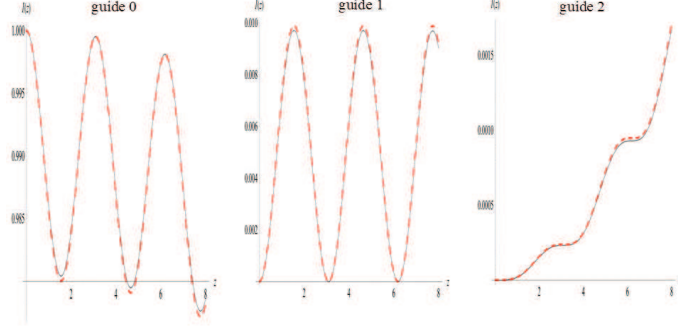


Figure 2: Comparison between the exact numerical solution (continuous black line) and the small rotation approximation solution (dashed red line) for $\omega = 1$ and $\alpha = 0.1$, and for the 3 first guides

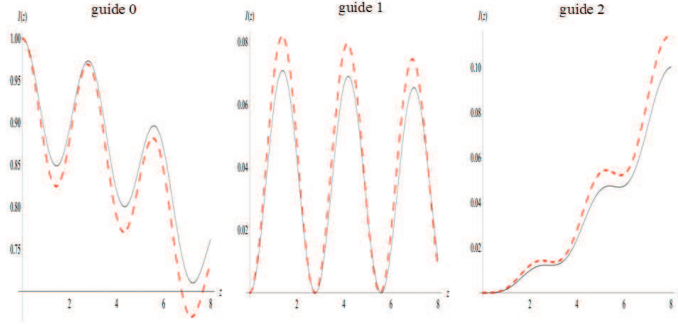


Figure 3: Comparison between the exact numerical solution (continuous black line) and the small rotation approximation solution (dashed red line) for $\omega = 1$ and $\alpha = 0.3$, and for the 3 first guides

explicit expressions for the operators \hat{R} and \hat{R}^\dagger ,

$$\hat{R} = e^{\frac{\alpha}{2\omega}(-1)^{\hat{n}}(\hat{V} + \hat{V}^\dagger)} = \sum_{j=-\infty}^{\infty} J_j\left(\frac{\alpha}{\omega}\right) [(-1)^{\hat{n}}\hat{V}]^j, \quad (29)$$

and

$$\hat{R}^\dagger = e^{-\frac{\alpha}{2\omega}(-1)^{\hat{n}}(\hat{V}+\hat{V}^\dagger)} = \sum_{\mu=-\infty}^{\infty} (-1)^\mu J_\mu\left(\frac{\alpha}{\omega}\right) \left[(-1)^{\hat{n}}\hat{V}\right]^\mu, \quad (30)$$

that after being substituted in Equation (28) give us

$$\begin{aligned} |\psi(z)\rangle &= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{\mu=-\infty}^{\infty} (-i)^k (-1)^\mu J_k\left(\frac{\alpha^2 z}{\omega}\right) J_j\left(\frac{\alpha}{\omega}\right) J_\mu\left(\frac{\alpha}{\omega}\right) \\ &\quad \hat{V}^{2k} \left[(-1)^{\hat{n}}\hat{V}\right]^\mu e^{-iz\left(\frac{\omega^2+\alpha^2}{\omega}\right)(-1)^{\hat{n}}} \left[(-1)^{\hat{n}}\right]^k \left[(-1)^{\hat{n}}\hat{V}\right]^j |\psi(0)\rangle. \end{aligned} \quad (31)$$

It is possible to show that

$$\left[(-1)^{\hat{n}}\hat{V}\right]^j |m\rangle = (-1)^{jm - \frac{j(j+1)}{2}} |m-j\rangle \quad (32)$$

for j positive and negative; so considering the initial condition $|\psi(0)\rangle = |m\rangle$, and after some algebra

$$\begin{aligned} u_n(z) &= \langle n|\psi(z)\rangle_m = (-1)^{\frac{m(m-1)-n(n-1)}{2}} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{k(m-j)} e^{-i(-1)^{m-j}\left(\frac{\omega^2+\alpha^2}{\omega}\right)z} \\ &\quad i^k J_k\left(\frac{\alpha^2 z}{\omega}\right) J_j\left(\frac{\alpha}{\omega}\right) J_{n-m+2k+j}\left(\frac{\alpha}{\omega}\right) \end{aligned}$$

is obtained.

If we consider that the "central" waveguide is shined ($m = 0$), we get

$$u_n(z) = (-1)^{\frac{n(n-1)}{2}} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{jk} e^{-i(-1)^j\left(\frac{\omega^2+\alpha^2}{\omega}\right)z} i^k J_k\left(\frac{\alpha^2 z}{\omega}\right) J_j\left(\frac{\alpha}{\omega}\right) J_{n+2k+j}\left(\frac{\alpha}{\omega}\right), \quad (33)$$

that is the final solution in this approximation.

Using that

$$\sum_{j=1}^{\infty} [J_j(x)]^2 = \frac{1}{2} \{1 - [J_0(x)]^2\}, \quad (34)$$

and that

$$\sum_{j=-\infty}^{\infty} J_j(x) J_{n+j}(x) = 0, \quad (35)$$

it is possible to show that the initial conditions are satisfied. It is also easy to verify that in the special case when $\alpha = 0$, we get the correct very well known trivial solution.

In the Figure 2, we compare the exact solution and the small rotation approximation for $\alpha = 0.1$ and in Figure 3 for $\alpha = 0.3$ giving still good accuracy even for larger α 's.

5 Conclusions

It has been shown how we can obtain, to a good degree of accuracy approximated solutions to the problem of light propagating in waveguide arrays. A Rayleigh-Schrödinger perturbation approach had been proposed previously by Rother, [10] that however gives rise to convergence problems. Here because of the method applied, i.e. the small rotation approach, these convergence problems are eliminated.

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